Uniform hypocoercivity and diffusive limit for Vlasov-Fokker-Planck

Joint work with Maxime Herda[1]

Pierre Gervais[2]

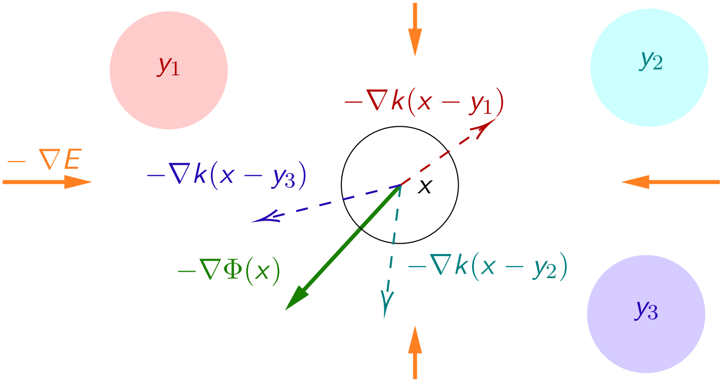
January 12, 2026

[1]INRIA, Université de Lille, RAPSODI team
[2]IMT, Université de Toulouse

Introduction

General setting

- Confining force $-\nabla E$
- $-\nabla k(z)$ = force at relative distance $z \Rightarrow$ mean force $-\nabla\Phi$
- Depending on the scale: diffusion, damping



The Vlasov-Fokker-Planck model

Random fluctuation and damping of velocity:

$$X_t'' = -X_t' - \nabla E - \nabla \Phi + \sqrt{2} dB_t$$

Vlasov-Fokker-Planck : phase-space density $F(t,x,v)$ of particles

$$\begin{cases} \partial_t F + v \cdot \nabla_x F - \nabla_x (E + \Phi) \cdot \nabla_v F = \nabla_v \cdot (vF + \nabla_v F) \\ \Phi(t,x) = \displaystyle\int_{\mathbb{R}^d} k(x-y) R(t,y) \mathrm{d}y \end{cases}$$

where $R(t,x) = \displaystyle\int_{\mathbb{R}^d} F(t,x,v)\mathrm{d}v$ is the macoscopic density.

The McKean-Vlasov model

Random fluctuation of position:

$$X_t' = -\nabla E - \nabla \Phi + \sqrt{2} dB_t$$

McKean-Vlasov : density $R(t,x)$ of particles

$$\begin{cases} \partial_t R = \Delta_x R + \nabla_x \cdot \left(R \nabla_x \left(E + \Phi\right)\right) \\ \Phi(t,x) = \displaystyle\int_{\mathbb{R}^d} k(x-y) R(t,y) \mathrm{d}y \end{cases}$$

VFP with time scale $\sim \varepsilon$, fluctuation/damping $\sim \frac{1}{\varepsilon}$:

$$F\left(t,x,v\right) \xrightarrow{\varepsilon \to 0} \frac{1}{\sqrt{2\pi}^d} R(t,x) e^{-|v|^2/2}$$

Our motivation:

1. When is the equilibrium (non-)unique?
2. When is it (un-)stable? Attractive?

Related questions:

1. What is the rate of convergence to equilibrium?
2. Regularizing effects?
3. How well does MKV approximates VFP?

Free energy for McKean-Vlasov (similar for VFP)

Free energy: $\mathcal{F}[R] = \int_{\mathbb{R}^d} R \log R \, \mathrm{d}x + \int_{\mathbb{R}^d} R \left(E + \frac{1}{2}\Phi \right) \mathrm{d}x$

$$\frac{\mathrm{d}\mathcal{F}[R]}{\mathrm{d}t} + \int_{\mathbb{R}^d} R \left| \nabla_x \log \left(e^{E+\Phi} R \right) \right|^2 \mathrm{d}x = \int_{\mathbb{R}^d} (\partial_t R) \Phi^{skew} \mathrm{d}x$$

with $\Phi^{skew} = k^{skew} * R$ induced by $k^{skew}(x) = \frac{k(x) - k(-x)}{2}$.

Consequence: k symmetric $\Rightarrow \mathcal{F}$ Lyapunov

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Consequence: k symmetric $\Rightarrow \mathcal{F}$ Lyapunov

There is always at least one equilibrium $R_\infty = \frac{1}{Z_\infty} e^{-E-\Phi_\infty}$, and

$$\left(\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}R}\left[R^1_\infty\right] - \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}R}\left[R^2_\infty\right] \right) \cdot \left(R^1_\infty - R^2_\infty\right) = 0\,.$$

Consequence: $\mathcal{F}$ strictly convex $\Rightarrow$ unique equilibrium

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Free energy: $\mathcal{F}[R] = \int_{\mathbb{R}^d} R \log R \, \mathrm{d}x + \int_{\mathbb{R}^d} R \left(E + \frac{1}{2} k * R \right) \mathrm{d}x$

$$\frac{\mathrm{d}\mathcal{F}[R]}{\mathrm{d}t} + \int_{\mathbb{R}^d} R \left| \nabla_x \log \left(e^{E+\Phi} R \right) \right|^2 \mathrm{d}x = \int_{\mathbb{R}^d} (\partial_t R) \Phi^{skew} \mathrm{d}x$$

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There is always at least one equilibrium $R_\infty = \frac{1}{Z_\infty} e^{-E-\Phi_\infty}$, and

$$\left(\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}R} \left[R_\infty^1 \right] - \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}R} \left[R_\infty^2 \right] \right) \cdot \left(R_\infty^1 - R_\infty^2 \right) = 0 \, .$$

Consequence: $\Re \widehat{k} \geq 0 \Rightarrow$ unique equilibrium

Examples: effect of the interactions on the dynamics

The ideal situation seems to be when k is symmetric and $\widehat{k} \geq 0$.

- Symmetric & positive: Coulomb : $k(x) = |x|^{2-d}$

  Unique globally **attractive** equilibrium for VFP: *Bouchut, Dolbeault (1995)*

Examples: effect of the interactions on the dynamics

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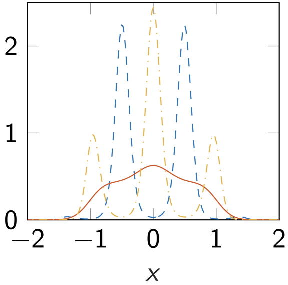
- Symmetric & positive: Coulomb : $k(x) = |x|^{2-d}$

And otherwise?

- Symmetric & ~~positive~~: Kuramoto, $k(x) = -I\cos(x)$ $(I \gg 1)$

  Carillo et Al. (2020), work in progress with Cesbron, Herda



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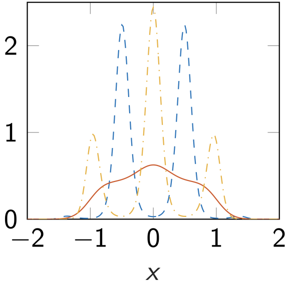
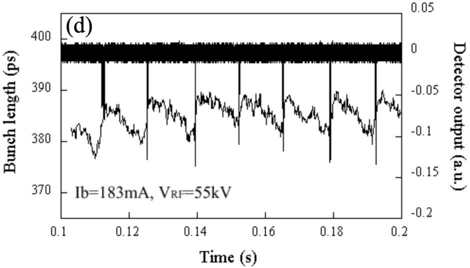
- ~~Symmetric~~: Synchrotron model

Roussel (PhD, 2014), Cai (2011)





Main results

We make the following assumptions:

1. The interaction term is a bounded operator

$$\|k * \rho\|_{L^p} + \|\nabla k * \rho\|_{L^q} \lesssim \|\rho\|_{L^1 \cap L^2}, \quad p, q \geq 2$$

2. The interactions are almost symmetric–positive

$$\begin{cases} \|k^{skew} * \rho\|_{L^p} + \|\nabla k^{skew} * \rho\|_{L^q} \leq \delta \|\rho\|_{L^1 \cap L^2} \\ \langle k * \rho, \rho \rangle_{L^2} = \left\langle \widehat{k}\widehat{\rho}, \widehat{\rho} \right\rangle_{L^2} \geq -\delta \|\rho\|^2_{L^1 \cap L^2} \end{cases} \qquad (0 \leq \delta \ll 1)$$

NB: the symmetric-positive part of k can be large!

3. E is "smooth" and "confines in a bounded region of space"

Main results

We make the following assumptions:

1. The interaction term is a bounded operator
2. The interactions are almost symmetric–positive
3. E is "smooth" and "confines in a bounded region of space"

Theorem

The steady state F_∞ is locally attractive and the flow regularizes:

$$\forall n, t \geq 0 \quad t^{\frac{3n}{2}} e^{\lambda t} \|F(t) - F_\infty\|_{\mathcal{H}^{s+n}} \leq C_n \|F(0) - F_\infty\|_{\mathcal{H}^s}$$

where $s > \frac{d}{q} - \frac{1}{3}$ and $\|F\|_{\mathcal{H}^s} = \|F/F_\infty\|_{H^s(F_\infty)}$.

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where $s > \frac{d}{q} - \frac{1}{3}$ *and* $\|F\|_{\mathcal{H}^s} \approx \|F\|_{H^s\left(F_\infty^{-1}\right)} + \||\nabla_x E|^s F\|_{L^2\left(F_\infty^{-1}\right)}$.

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Novelties of the Cauchy theory

- Taking into account non-symmetric interactions
  Cesbron, Herda (2024): weakly nonlinear regime for synchrotron
- No size restriction for symmetric-positive interactions
  Addala, Dolbeault, Li, Tayeb (2020),*Toshpulatov (2025)*: Poisson
- Working in H^s spaces with $s \in [0, \infty)$
  Allows for weakly regularizing interactions ($q = 2$), higher order regularization
- For Poisson interactions: strongly nonlinear regime in $H_x^{\frac{1}{6}+} L_v^2$
  Hérau, Thomann (2016): weakly nonlinear regime in $H_{x,v}^{\frac{1}{2}+}$

Main results

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Theorem

The scaled density $F^\varepsilon(t,x,v) = F\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, v\right)$ is the sum of

- *its limit $R(t,x)M(v)$ where R solves MKV with*

$$R(0,x) = \int F^\varepsilon(0,x,v)\mathrm{d}v\,, \quad \|R(t)-R_\infty\|_{\mathcal{H}^s} \lesssim e^{-\lambda t}\|R(0)-R_\infty\|_{\mathcal{H}^s}$$

- *an initial layer controlled by $e^{-\lambda t/\varepsilon^2}\|F^\varepsilon(0) - R(0)M\|_{\mathcal{H}^s}$*
- *an error term controlled by $\varepsilon e^{-\lambda t}\|F^\varepsilon(0)\|_{\mathcal{H}^{s+1}}$*

Main results

We make the following assumptions:

1. The interaction term is a bounded operator
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3. E is “smooth” and “confines in a bounded region of space”

Novelties of the diffusive limit

- Uniform rate of convergence in ε and t by energy methods
  Blaustein (2023), *Herda, Rodrigues (2018)*: integral rate of convergence
  Zhong (2022): spectral method
- Loss of only 1 derivative for convergence $\mathcal{O}(\varepsilon)$
  Zhong (2022): 2 derivatives

Cauchy theory close to equilibrium

Perturbation: $(F, \Phi, R) = (F_\infty, \Phi_\infty, R_\infty) + (f, \phi, \rho)$

Functional framework: hilbert space $L^2(F_\infty^{-1}\mathrm{d}x\mathrm{d}v)$ where

$$F_\infty(x, v) = \frac{1}{Z_\infty} e^{-\frac{|v|^2}{2} - E_\infty(x)}\,, \qquad E_\infty(x) = E(x) + \Phi_\infty(x)$$

We denote the velocity vector fields and "maxwellian" (gaussian)

$$D_v = \nabla_v - v\,, \qquad D_v^* = -\nabla_v\,, \qquad M(v) = \frac{1}{(2\pi)^{d/2}} e^{-|v|^2/2}\,.$$

Nice structure of the equation:

$$\partial_t f + Tf + D_v^* \cdot D_v f = -v \cdot F_\infty \nabla_x \phi + D_v^* \cdot (f \nabla_x \phi)$$

with $Tf = v \cdot \nabla_x f - \nabla_x \Phi_\infty \cdot \nabla_v f = -T^*$.

Difficulties of the linear equation

Linearized perturbed equation:

$$\partial_t f + Tf + D_v^* \cdot D_v f = -v \cdot F_\infty \nabla_x \phi$$

Naive estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2 + \lambda \|D_v f(t)\|^2 \lesssim \|\nabla_x \phi\|^2$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2 + \lambda\|f(t) - \rho(t)M\|^2 \lesssim \|\nabla_x \phi\|^2$$

(Gaussian Poincaré in v & $\rho(t,x) = \int f(t,x,v)\mathrm{d}v$)

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Problem 1

How to control ρ?

Problem 2

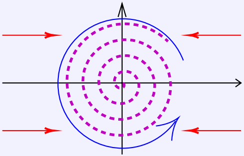
Even in the ideal case of k sym-pos, $\nabla_x \phi$ appears as a bad term.

Problem 1: how to control ρ? Show hypocoercivity!

2D toy-model for hypocoercivity

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} u$$

$$\text{Eigenvalues} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

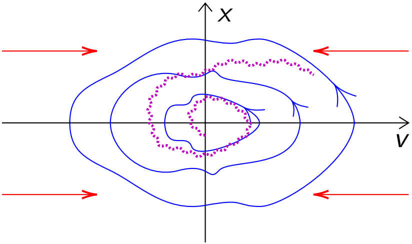


- Incomplete energy estimate $\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2 = -2u_2^2(t) \not\Rightarrow$ decay $\mathcal{O}\left(e^{-\frac{t}{2}}\right)$
- Introduce the equivalent (squared) norm ($|\eta| < 1$)

$$H(u) = u_1^2 + u_2^2 + 2\eta u_1 u_2 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}H(u(t)) + H(u(t)) \leq 0\,.$$

Problem 1: how to control ρ? Show hypocoercivity!

Focusing on transport and diffusion: $\partial_t f + Tf + D_v^* \cdot D_v f = 0$



Transport along level lines of $\frac{|v|^2}{2} + E_\infty(x)$

$+$ Diffusion in v

$=$ Diffusion in (x, v)

Most strategies rely on energy estimates and the identities

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Af, f\rangle = \langle [A, T]f, f\rangle + \dots \quad \text{and} \quad [\nabla_v, T] = \nabla_x$$

Example: with $A = \nabla_x \nabla_v$ we retrieve using (x, v)-Poincaré

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Af, f\rangle = -\|\nabla_x f\|^2 + \dots \lesssim -\|f\|^2 + \|\nabla_v f\|^2 + \dots$$

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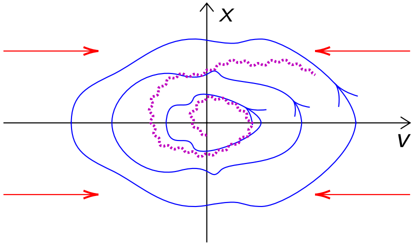

Transport along level lines of $\frac{|v|^2}{2} + E_\infty(x)$
$+$ Diffusion in v
$=$ Diffusion in (x, v)

Combination for $|\alpha| \leq 1$ of $\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha_{x,v} f\|^2 + \|\partial^\alpha_{x,v}\nabla_v f\|^2 \lesssim \dots \\ \frac{\mathrm{d}}{\mathrm{d}t}\langle Af, f\rangle + \|\nabla_x f\|^2 \lesssim \dots \end{cases}$

allows to prove $\|f(t)\|^2 + t^3\|\nabla_x f(t)\|^2 + t\|\nabla_v f\|^2 \lesssim e^{-\lambda t}\|f(0)\|^2$.

Problem 2: if k is sym-pos, $\nabla_x \phi$ must not be a bad term

MKV Reminder: if k is symmetric, free energy $\mathcal{F}$ is Lyapunov
$\Rightarrow$ good candidate for a norm

$$\mathcal{F}[R_\infty + \rho] = \mathcal{F}[R_\infty] + \int_{\mathbb{R}^d} \left(|\rho|^2 R_\infty^{-1} + \rho\phi^{sym}\right) \mathrm{d}x + \mathcal{O}(\|\rho\|^3)$$

$$\left(\phi^{sym} = k^{sym} * \rho \quad \& \quad k^{sym}(x) = \tfrac{k(x)+k(-x)}{2} \right)$$

Same space (equivalent norm) but **new hilbert structure**:

Addala, Dolbeault, Li, Tayeb (2020)

$$|||f|||^2 = \|f\|^2 + \int_{\mathbb{R}^d} \rho\phi^{sym}\mathrm{d}x \approx \|f\|^2$$

$$\begin{pmatrix} \text{boundedness of } k * (\cdot) \Rightarrow \int_{\mathbb{R}^d} \rho\phi^{sym}\mathrm{d}x \lesssim \|\rho\|^2 \\ \\ \Re\widehat{k} \geq 0 \Rightarrow \int_{\mathbb{R}^d} \rho\phi^{sym}\mathrm{d}x = \int_{\mathbb{R}^d} \Re\widehat{k}|\rho|^2\mathrm{d}\xi \geq 0 \end{pmatrix}$$

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New **(skew-adjoint!)** transport: $\mathcal{T} := T - vF_\infty \cdot \nabla_x \phi^{sym} = -\mathcal{T}^*$

$$\partial_t f + \mathcal{T} f + D_v^* \cdot D_v f = v \cdot F_\infty \nabla_x \phi^{skew}$$

Conclusion: only $v \cdot F_\infty \nabla_x \phi^{skew}$ needs to be small.

Remaining problem: the nonlinearity

Equation with source χ instead of nonlinearity $f\nabla_x\phi$:

$$\partial_t f + \mathcal{T} f + D_v^* \cdot D_v f = -v \cdot F_\infty \nabla_x \phi^{skew} + D_v^* \cdot \chi$$

Proposition:

1. $\forall s \geq 0$, we have $\|f\|_{\mathscr{X}} \lesssim \|f(0)\|_{H_x^s L_v^2} + \|\chi\|_{\mathscr{Y}}$ (previous slides).
2. $\forall s > \frac{d}{q} - \frac{1}{3}$, we have $\|f\nabla_x\phi\|_{\mathscr{Y}} \lesssim \|f\|_{\mathscr{X}}^2$.

NB: $\mathscr{X}$ encodes exponential decay & regularization (see Slide 18)

Consequence: Picard iteration $\Rightarrow$ unique solution in $\mathscr{X}$.

Rough idea of the nonlinear proof: a toy model for VFP

Heuristic: Hypoellipticity in VFP $\Leftrightarrow \nabla_v \simeq \nabla_x^{\frac{1}{3}}$

$$\partial_t u + (1-\Delta)^{\frac{1}{3}} u = \nabla^{\frac{1}{3}}(u \nabla k * u) \qquad \text{(VFP toy model)}$$

Energy estimate in H^s:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_s^2 + \lambda \|u\|_{s+\frac{1}{3}}^2 \lesssim \|u \nabla k * u\|_s^2$$

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(Kato-Ponce with $\frac{1}{r} + \frac{1}{q} = \frac{1}{2}$ & " $W^{\frac{d}{q}+\delta,q} \hookrightarrow L^\infty$ ")

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(Boundedness " $\nabla k * (\cdot) : H^\sigma \to W^{\sigma,q}$ ")

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We want s such that $s + \frac{1}{3} > \frac{d}{q}$ and $H^{s+\frac{1}{3}} \hookrightarrow L^r$.

Same condition: $s > \frac{d}{q} - \frac{1}{3}$.

Diffusive limit

The diffusive scaling

We consider the (perturbation) equation in diffusive scaling:

$$\varepsilon^2 \partial_t f^\varepsilon + \varepsilon T f^\varepsilon + D_v^* \cdot D_v f^\varepsilon = \varepsilon v \cdot F_\infty \nabla_x \phi^\varepsilon + \varepsilon D_v^* \cdot (f^\varepsilon \nabla_x \phi^\varepsilon)$$

All previous estimates still hold uniformly in $\varepsilon \in (0,1)$:

- Decay $\mathcal{O}\left(e^{-\lambda t}\right)$
- Regularization $\mathcal{O}\left(t^{-3}\right)$

Naive energy estimate: $\displaystyle\int_0^\infty e^{2\lambda t} \|D_v f^\varepsilon(t)\|^2 \mathrm{d}t \lesssim \varepsilon^2 \|f^\varepsilon(0)\|^2$

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Our goal

Show $f^\varepsilon(t,x,v) \xrightarrow{\varepsilon \to 0} \rho^0(t,x) M(v)$ where $R = R_\infty + \rho^0$ solves

$$\partial_t R = \Delta_x R + \nabla_x \cdot (R \nabla_x (E + \Phi)) \, .$$

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Strategy

(1) Macroscopic behavior: $\rho^\varepsilon = \rho^0 + \mathcal{O}(\varepsilon)$
(2) Well prepared case: $f^\varepsilon_{wp} = \rho^\varepsilon + \mathcal{O}(\varepsilon)$
(3) Ill prepared case: $f^\varepsilon = f^\varepsilon_{wp} + \textit{explicit initial layer} + \mathcal{O}(\varepsilon)$

Conclusion: $f^\varepsilon = \rho^0 + \text{explicit initial layer} + \mathcal{O}(\varepsilon)$

Step 1: Macroscopic behavior

Equation for the density ρ^ε (and the flux $j^\varepsilon = \int v f^\varepsilon \mathrm{d}v$)

$$\begin{aligned}\partial_t(\rho^\varepsilon - \varepsilon\nabla_x \cdot j^\varepsilon) = &\text{pertubed MKV}(\rho^\varepsilon - \varepsilon\nabla_x \cdot j^\varepsilon) \\ &+ \mathcal{O}\left(\nabla_x^2(f^\varepsilon - \rho^\varepsilon M)\right) + \mathcal{O}\left(\varepsilon\nabla_x^3(f^\varepsilon - \rho^\varepsilon M)\right)\end{aligned}$$

Comparing with the solution ρ^0 of perturbed MKV and using

$$\int e^{2\lambda t}\|f^\varepsilon(t) - \rho^\varepsilon(t)M\|_{\mathcal{H}^s}^2 \mathrm{d}t \lesssim \varepsilon^2\|f^\varepsilon(0)\|_{\mathcal{H}^s}$$

we obtain (for small initial data)

$$\sup_{t\geq 0} e^{\lambda t}\|\rho^\varepsilon - \rho^0\|_{\mathcal{H}^s} \lesssim \varepsilon\|f^\varepsilon(0)\|_{\mathcal{H}^{s+1}}$$

See Slide 19.

Step 2: Microscopic behavior in the well prepared case

Assume the initial data is well-prepared:

$$f^\varepsilon(0,x,v) = \rho^0(0,x)M(v)$$

Take the microscopic part of f^ε:

$$\partial_t(f^\varepsilon - \rho^\varepsilon M) + \frac{1}{\varepsilon^2} D_v^* \cdot D_v(f^\varepsilon - \rho^\varepsilon M) = \mathcal{O}\left(\frac{1}{\varepsilon}\nabla_x f^\varepsilon\right) + \dots$$

Energy estimate (zero at $t = 0$):

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f^\varepsilon(t) - \rho^\varepsilon(t)M\|^2_{\mathcal{H}^s} + \frac{\lambda}{\varepsilon^2}\|f^\varepsilon(t) - \rho^\varepsilon(t)M\|^2_{\mathcal{H}^s} \lesssim e^{-\lambda t}\|f^\varepsilon(0)\|_{\mathcal{H}^{s+1}}$$

which yields $e^{\lambda t}\|f^\varepsilon(t) - \rho^\varepsilon(t)M\|_{\mathcal{H}^s} \lesssim \varepsilon\|f^\varepsilon(0)\|_{\mathcal{H}^{s+1}}$.

Step 3: The ill prepared case

Initial layer ansatz $\boldsymbol{f_{il}^{\varepsilon}}$:

$$\begin{cases} \partial_t f_{il}^{\varepsilon} = \text{pertubed VFP}^{\varepsilon}\left(f_{il}^{\varepsilon}\right) - \dfrac{\mathbf{1}}{\boldsymbol{\varepsilon^2}}\boldsymbol{\rho[f_{il}^{\varepsilon}]}\,, \\ \\ f_{il}^{\varepsilon}(0) = f^{\varepsilon}(0) - \rho^{\varepsilon}(0)M\,, \end{cases}$$

which yields $\|f^{\varepsilon}(t)\|_{\mathcal{H}^s} \lesssim e^{-\lambda t/\varepsilon^2}\|f^{\varepsilon}(0) - \rho^{\varepsilon}(0)M\|_{\mathcal{H}^s}$.

Equation for $\boldsymbol{u^{\varepsilon} = f^{\varepsilon} - \left(f_{wp}^{\varepsilon} + f_{il}^{\varepsilon}\right)}$:

$$\partial_t u^{\varepsilon} + \frac{1}{\varepsilon^2}(D_v^* \cdot D_v)u^{\varepsilon} + \frac{1}{\varepsilon}Tu^{\varepsilon} = \frac{\mathbf{1}}{\boldsymbol{\varepsilon^2}}\boldsymbol{\rho[f_{il}^{\varepsilon}]} + \dots\,, \quad u^{\varepsilon}(0) = 0$$

which yields $e^{\lambda t}\|u^{\varepsilon}(t)\|_{\mathcal{H}^s} \lesssim \varepsilon\|f^{\varepsilon}(0)\|_{\mathcal{H}^{s+1}}$.

Bibliography

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Works at the basis of the talk:

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Norms for the linear and nonlinear flow of VFP

Norm for controlled quantities:

$$\|f\|_{\mathscr{X}}^2 = \sup_{t\geq 0} e^{2\lambda t}\left(\|f(t)\|_{H_x^s L_v^2}^2 + t^3\|f(t)\|_{H_x^{s+1} L_v^2}^2 + t\|f(t)\|_{H_x^s H_v^1}^2\right) + \dots$$

Norm for source terms:

$$\|f\|_{\mathscr{Y}}^2 = \int_0^\infty e^{2\lambda t}\left(\|f(t)\|_{H_x^s L_v^2}^2 + t^3\|f(t)\|_{H_x^{s+1} L_v^2}^2 + t\|f(t)\|_{H_x^s H_v^1}^2\right)\mathrm{d}t$$

Macroscopic behavior

Equation for the density ρ^ε (and the flux $j^\varepsilon = \int v f^\varepsilon \mathrm{d}v$)

$$\begin{cases} \partial_t \rho^\varepsilon + \frac{1}{\varepsilon}\nabla_x \cdot j^\varepsilon = 0 \\ \\ \dfrac{j^\varepsilon}{\varepsilon} = -\left(\nabla_x \rho^\varepsilon + \rho^\varepsilon \nabla_x (E_\infty + \phi^\varepsilon) + R_\infty \nabla_x \phi^\varepsilon\right) - \varepsilon \partial_t j^\varepsilon \\ \qquad\qquad + \mathcal{O}\left(\nabla_x (f^\varepsilon - \rho^\varepsilon M)\right) \end{cases}$$

Macroscopic behavior

Equation for the density ρ^ε

$$\partial_t \rho^\varepsilon = \text{pertubed MKV}(\rho^\varepsilon) + \varepsilon \partial_t \nabla_x \cdot j^\varepsilon + \mathcal{O}\left(\nabla_x^2 (f^\varepsilon - \rho^\varepsilon M)\right)$$

Macroscopic behavior

Equation for the density ρ^ε

$$\begin{aligned}\partial_t(\rho^\varepsilon - \varepsilon\nabla_x \cdot j^\varepsilon) =& \text{pertubed MKV}(\rho^\varepsilon - \varepsilon\nabla_x \cdot j^\varepsilon) \\ &+ \mathcal{O}\left(\nabla_x^2(f^\varepsilon - \rho^\varepsilon M)\right) + \mathcal{O}\left(\varepsilon\nabla_x^3(f^\varepsilon - \rho^\varepsilon M)\right)\end{aligned}$$

Macroscopic behavior

Equation for the density ρ^ε

$$\partial_t(\rho^\varepsilon - \varepsilon\nabla_x \cdot j^\varepsilon) = \text{pertubed MKV}(\rho^\varepsilon - \varepsilon\nabla_x \cdot j^\varepsilon) + \mathcal{O}\left(\nabla_x^2(f^\varepsilon - \rho^\varepsilon M)\right) + \mathcal{O}\left(\varepsilon\nabla_x^3(f^\varepsilon - \rho^\varepsilon M)\right)$$

Using the naive energy estimate

$$\int e^{2\lambda t}\|f^\varepsilon(t) - \rho^\varepsilon(t)M\|_{\mathcal{H}^s}^2 \mathrm{d}t \lesssim \varepsilon^2\|f^\varepsilon(0)\|_{\mathcal{H}^s}^2$$

we establish that

$$e^{\lambda t}\left\|\rho^0 - (\rho^\varepsilon - \varepsilon\nabla_x \cdot j^\varepsilon)\right\|_{\mathcal{H}^s} \lesssim \varepsilon\|f^\varepsilon(0)\|_{\mathcal{H}^{s+1}} + \varepsilon^2\|f^\varepsilon(0)\|_{\mathcal{H}^{s+2}}$$

NB: Regularizing term Δ_x in MKV $\Rightarrow$ reduced loss of regularity.

Macroscopic behavior

Equation for the density ρ^ε

$$\begin{aligned}\partial_t(\rho^\varepsilon - \varepsilon\nabla_x \cdot j^\varepsilon) =& \text{pertubed MKV}(\rho^\varepsilon - \varepsilon\nabla_x \cdot j^\varepsilon) \\ &+ \mathcal{O}\left(\nabla_x^2(f^\varepsilon - \rho^\varepsilon M)\right) + \mathcal{O}\left(\varepsilon\nabla_x^3(f^\varepsilon - \rho^\varepsilon M)\right)\end{aligned}$$

Using the naive energy estimate

$$\int e^{2\lambda t}\|f^\varepsilon(t) - \rho^\varepsilon(t)M\|^2_{\mathcal{H}^s}\mathrm{d}t \lesssim \varepsilon^2\|f^\varepsilon(0)\|^2_{\mathcal{H}^s}$$

we establish that

$$e^{\lambda t}\left\|\rho^0 - \rho^\varepsilon\right\|_{\mathcal{H}^s} \lesssim \varepsilon\|f^\varepsilon(0)\|_{\mathcal{H}^{s+1}} + \varepsilon^2\|f^\varepsilon(0)\|_{\mathcal{H}^{s+2}}$$

NB: Use that $\|\nabla \cdot j^\varepsilon\|_{\mathcal{H}^s} \lesssim \|f^\varepsilon\|_{\mathcal{H}^{s+1}}$.

Macroscopic behavior

Equation for the density ρ^ε

$$\partial_t(\rho^\varepsilon - \varepsilon\nabla_x \cdot j^\varepsilon) = \text{pertubed MKV}(\rho^\varepsilon - \varepsilon\nabla_x \cdot j^\varepsilon) + \mathcal{O}\left(\nabla_x^2(f^\varepsilon - \rho^\varepsilon M)\right) + \mathcal{O}\left(\varepsilon\nabla_x^3(f^\varepsilon - \rho^\varepsilon M)\right)$$

Using the naive energy estimate

$$\int e^{2\lambda t}\|f^\varepsilon(t) - \rho^\varepsilon(t)M\|^2_{\mathcal{H}^s}\mathrm{d}t \lesssim \varepsilon^2\|f^\varepsilon(0)\|^2_{\mathcal{H}^s}$$

we establish that

$$e^{\lambda t}\left\|\rho^0 - \rho^\varepsilon\right\|_{\mathcal{H}^s} \lesssim \varepsilon\|f^\varepsilon(0)\|_{\mathcal{H}^{s+1}}$$

NB: Regularization of the initial data
$\|\widetilde{f}^\varepsilon(0)\|_{\mathcal{H}^{s+2}} \lesssim \frac{1}{\varepsilon}\|f^\varepsilon(0)\|_{\mathcal{H}^{s+1}}$.